**Limit theorems for multiscale stochastic
dynamical systems**
Longjie Xie
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16th Workshop on Markov Processes and Related Top
Changsha, July 12-16, 2021
exie (JSNU)
Nultiscale stochastic systems Limit theorems for multiscale stochastic dynamical systems

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The 16th Workshop on Markov Processes and Related Topics Changsha, July 12-16, 2021

[Background](#page-2-0)

- Experiment of the New York of Averaging principle - functional LLN
- [Normal deviations functional CLT](#page-16-0)

[Main results](#page-19-0)

- **•** [Functional LLN](#page-19-0)
- **[Functional CLT: Case 0](#page-29-0)**
- **[Functional CLT: Case 1](#page-35-0)**
- [Functional CLT: Case 2](#page-39-0)

Consider the two-time-scales stochastic system:

$$
dY_t^{\varepsilon} = F(X_{t/\varepsilon}, Y_t^{\varepsilon})dt + dW_t, \quad Y_0 = y \in \mathbb{R}^d \tag{1.1}
$$

OUN[D](#page-0-0)ER A Veraging principle

the two-time-scales stochastic system:
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 $= (X_t)_{t \geqslant 0}$ is an ergodic Markov process possessing a unique

measure $\mu(dx)$, and $0 < \varepsilon \$ where $X = (X_t)_{t \geq 0}$ is an ergodic Markov process possessing a unique invariant measure $\mu(dx)$, and $0 < \varepsilon \ll 1$ represents the separation of time scales.

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= $(X_t)_{t\geqslant0}$ is an ergodic Markov process possessing a unique measure $\mu(dx)$, and 0 < ε ≪ 1 rep where $X = (X_t)_{t \geq 0}$ is an ergodic Markov process possessing a unique invariant measure $\mu(dx)$, and $0 < \varepsilon \ll 1$ represents the separation of time scales.

- \triangleright Y_{t}^{ε} (slow variable): mathematical model for a phenomenon appearing at the natural time scale;
- $\triangleright X_{t/\varepsilon}$ (fast variable): fast random environment/effects at a faster time scale (with time order $1/\varepsilon$).

Background - Averaging principle

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 $= (X_t)_t ≥ 0$ is an ergodic Markov process possessing a unique measure $μ(dx)$, and $0 < ε \ll 1$ represents the separ Usually, the system (1.1) is difficult to deal with due to the two widely separated time scales. Thus a simplified equation which governs the evolution of the system over a long time scale (as $\varepsilon \to 0$) is highly desirable.

Consider the two-time-scales stochastic system:

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dY_t^{\varepsilon} = F(X_{t/\varepsilon}, Y_t^{\varepsilon})dt + dW_t, \quad Y_0 = y \in \mathbb{R}^d \tag{1.1}
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Intuitively,

$$
X_{t/\varepsilon} \Rightarrow \mu(\mathrm{d} x) \quad \text{as} \quad \varepsilon \to 0.
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r averaging the coefficient with respect to the fast Thus, by averaging the coefficient with respect to the fast variable, the slow part Y_{t}^{ε} will converge to \bar{Y}_{t} , where

$$
\mathrm{d}\bar{Y}_t = \bar{F}(\bar{Y}_t)\mathrm{d}t + \mathrm{d}W_t, \quad Y_0 = y \in \mathbb{R}^d
$$

with

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\bar{F}(y) := \int_{\mathbb{R}^d} F(x, y) \mu(\mathrm{d} x).
$$

Background - Averaging principle

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y,
 $X_{t/ε} ⇒ μ(dx)$ as $ε → 0$.

v averaging the coefficient with respect to the fast variable,
 $Y_t^ε$ will conve This theory, known as the averaging principle, was first developed by Bogolyubov (1937) for ODEs and extended to the SDEs by Khasminskii (1966). Ω

Consider the following fast-slow stochastic system in $\mathbb{R}^{d_1+d_2}$:

ound - Averaging principle

\nthe following fast-slow stochastic system in
$$
\mathbb{R}^{d_1+d_2}
$$
:

\n
$$
\begin{cases}\n\mathrm{d}X_t^{\varepsilon} = \varepsilon^{-1}b(X_t^{\varepsilon}, Y_t^{\varepsilon})\mathrm{d}t + \varepsilon^{-1/2}\sigma(X_t^{\varepsilon}, Y_t^{\varepsilon})\mathrm{d}W_t^1, \\
\mathrm{d}Y_t^{\varepsilon} = F(X_t^{\varepsilon}, Y_t^{\varepsilon})\mathrm{d}t + G(Y_t^{\varepsilon})\mathrm{d}W_t^2, \\
X_0^{\varepsilon} = x \in \mathbb{R}^{d_1}, \quad Y_0^{\varepsilon} = y \in \mathbb{R}^{d_2}, \\
<\varepsilon \ll 1 \text{ is a small parameter.}\n\end{cases}
$$
\nand parameter.

\n2. (JSWU)

\nMultiscale stochastic systems

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where $0 < \varepsilon \ll 1$ is a small parameter.

Consider the following fast-slow stochastic system in $\mathbb{R}^{d_1+d_2}$:

[D](#page-0-0)raft dX ε ^t = ε −1 b(X ε t , Y ε t)dt + ε [−]1/2σ(X ε t , Y ε t)dW ¹ t , dY ε ^t = F(X ε t , Y ε t)dt + G(Y ε t)dW ² t , X ε ⁰ = x ∈ R d1 , Y ε ⁰ = y ∈ R d2 , (1.2)

where $0 < \varepsilon \ll 1$ is a small parameter.

The intuitive idea for deriving a simplified equation for (1.2) is based on the observation that:

 \circ during the fast transients, the slow variable remains "constant";

 \Diamond by the time its changes become noticeable, the fast variable has almost reached its quasi-steady state.

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blue that interest and the scaling $t \mapsto \varepsilon t$ **, the process** $\tilde{X}_t^{\varepsilon} := X_{\varepsilon t}^{\varepsilon}$ **satis
** $\mathrm{d}\tilde{X}_t^{\varepsilon} = b(\tilde{X}_t^{\varepsilon}, Y_{\varepsilon t}^{\varepsilon})\mathrm{d}t + \sigma(\tilde{X}_t^{\varepsilon}, Y_{\varepsilon t}^{\varepsilon})\mathrm{d}\tilde{W}_t^1,$ **
 \mathcal{U}_t^1 := \varepsilon^{-1/2} W_{** \diamond With the natural time scaling $t\mapsto \varepsilon t$, the process $\tilde X^\varepsilon_t:=X^\varepsilon_{\varepsilon t}$ satisfies

$$
\mathrm{d} \tilde{X}_t^\varepsilon = b(\tilde{X}_t^\varepsilon, Y_{\varepsilon t}^\varepsilon) \mathrm{d} t + \sigma(\tilde{X}_t^\varepsilon, Y_{\varepsilon t}^\varepsilon) \mathrm{d} \tilde{W}_t^1,
$$

where $\tilde{W}_t^1 := \varepsilon^{-1/2} W_{\varepsilon t}^1$ is a new BM.

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where $\tilde{W}_t^1 := \varepsilon^{-1/2} W_{\varepsilon t}^1$ is a new BM.

[D](#page-0-0)raft Thus we need to consider the auxiliary process $X_t^{\mathcal{Y}}$ which satisfies the frozen equation

$$
dX_t^y = b(X_t^y, y)dt + \sigma(X_t^y, y)dW_t^1, \quad X_0^y = x \in \mathbb{R}^{d_1}.
$$

Under certain recurrence conditions, the process $X_t^{\mathcal{Y}}$ process a unique invariant measure $\mu^y(\text{d} x)$.

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by averaging the coefficients with respect to parameter in faith slow part Y_t^{ε} will converge to \tilde{Y}_t , where
 $\mathrm{d}\, \tilde{Y}_t = \bar{F}(\tilde{Y}_t) \mathrm{d}t + G(\tilde{Y}_t) \mathrm{d}W_t^2$
 $\bar{F}(y) := \int_{\mathbb{R}^{d_1}} F(x, y) \mu^y(\mathrm{d}x).$
 $\$ \circ Then by averaging the coefficients with respect to parameter in fast variable, the slow part Y_{t}^{ε} will converge to \bar{Y}_{t} , where

$$
\mathrm{d}\bar{Y}_t = \bar{F}(\bar{Y}_t)\mathrm{d}t + G(\bar{Y}_t)\mathrm{d}W_t^2
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$$
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$$

Background - Averaging principle

Strong convergence:

$$
\begin{array}{ll}\n\text{count} - \text{Average:} \\
\text{sup} & \mathbb{E}|Y_t^{\varepsilon} - \bar{Y}_t| \leqslant C_T \, \varepsilon^{1/2}.\n\end{array}
$$
\n
$$
\text{curve:} \quad \begin{array}{ll}\n\text{sup} & \mathbb{E}|Y_t^{\varepsilon} - \bar{Y}_t| \leqslant C_T \, \varepsilon^{1/2}.\n\end{array}
$$
\n
$$
\text{Fole} \quad \text{sum} \quad \begin{array}{ll}\n\text{Fole} & \text{for } t \leqslant t \leqslant 0 \\
\text{Fole} & \text{for } t \leqslant t \leqslant t \leqslant 0 \\
\text{Fole} & \text{for } t \leqslant t \leqslant t \leqslant 0\n\end{array}
$$

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 \triangleright Strong convergence:

$$
\sup_{t\in[0,T]}\mathbb{E}|Y_{t}^{\varepsilon}-\bar{Y}_{t}|\leqslant C_{\mathcal{T}}\,\varepsilon^{1/2}.
$$

History results:

ound - Averaging principle
 $\sup_{t \in [0,T]} \mathbb{E}|Y_t^{\varepsilon} - \bar{Y}_t| \leqslant C_T \, \varepsilon^{1/2}.$

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i \bullet Freidlin and Wentcell (1998), Pavliotis and Stuart (2008), \cdots . Condition: all the coefficients are Lipschitz continuous.
- If G depends on the fast variable x, then the strong convergence may not hold.
- The convergence rate is important for numerical schemes (called HMM) for multiscale systems (see e.g. [E. etc, 2005, CPAM]).

However, the time scale separation is never infinite in reality.

ound - Normal deviation

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I but positive ε , the process Y_t^{ε} will experience fluctuations

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Ing order, these fluctuations can be ca To leading order, these fluctuations can be captured by characterizing the asymptotic behavior of the normalized difference

$$
Z_t^{\varepsilon} := \frac{Y_t^{\varepsilon} - \bar{Y}_t}{\sqrt{\varepsilon}} = \frac{1}{\sqrt{\varepsilon}} \int_0^t \left[F(X_s^{\varepsilon}, Y_s^{\varepsilon}) - \bar{F}(\bar{Y}_s) \right] \mathrm{d}s
$$

as ε tends to 0.

When $G \equiv \mathbb{I}_{d_2}$, the deviation process Z_t^ε is known to converge weakly to \bar{Z}_t with

$$
\mathrm{d}\bar{Z}_t = \nabla_y \bar{F}(\bar{Y}_t) \bar{Z}_t \mathrm{d}t + \zeta(\bar{Y}_t) \mathrm{d}\tilde{W}_t,
$$

where \tilde{W}_t is another standard Brownian motion, and the new diffusion coefficient is given by

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\dot{Z}_t with	$d\bar{Z}_t = \nabla_y \bar{F}(\bar{Y}_t) \bar{Z}_t dt + \zeta(\bar{Y}_t) d\tilde{W}_t$, where \tilde{W}_t is another standard Brownian motion, and the new diffusion
$\zeta(y) := \sqrt{\int_0^\infty \int_{\mathbb{R}^{d_1}} \mathbb{E}\big[F(X_t^y(x), y) - \bar{F}(y)\big] \big[F(x, y) - \bar{F}(y)\big]^* \mu^y(\mathrm{d}x) \mathrm{d}t}$.	
such result, also known as the Gaussian approximation, is an analogue of	
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Longie Xie (JSNU)	Multiscale stochastic systems

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\begin{cases}\ndX_t^{\varepsilon} = \alpha_{\varepsilon}^{-2}b(X_t^{\varepsilon}, Y_t^{\varepsilon})dt + \beta_{\varepsilon}^{-1}c(X_t^{\varepsilon}, Y_t^{\varepsilon})dt + \alpha_{\varepsilon}^{-1}\sigma(X_t^{\varepsilon}, Y_t^{\varepsilon})dW_t^1, \\
dY_t^{\varepsilon} = F(X_t^{\varepsilon}, Y_t^{\varepsilon})dt + \gamma_{\varepsilon}^{-1}H(X_t^{\varepsilon}, Y_t^{\varepsilon})dt + G(Y_t^{\varepsilon})dW_t^2, \\
X_0^{\varepsilon} = x \in \mathbb{R}^{d_1}, \quad Y_0^{\varepsilon} = y \in \mathbb{R}^{d_2}, \\
\text{where the small parameters } \alpha_{\varepsilon}, \beta_{\varepsilon}, \gamma_{\varepsilon} \to 0 \text{ as } \varepsilon \to 0. \tag{2.3}
$$
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\nMultiscale stochastic systems

\nJuly 15, 2021, 10, 10, 11.5, 2021, 2021, 2021, 2022, 2021, 202

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\mathrm{d}Y_t^{\varepsilon} = F(X_t^{\varepsilon}, Y_t^{\varepsilon})\mathrm{d}t + \gamma_{\varepsilon}^{-1}H(X_t^{\varepsilon}, Y_t^{\varepsilon})\mathrm{d}t + G(Y_t^{\varepsilon})\mathrm{d}W_t^2, \\
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\nwhere exist two time scales in the fast motion X_t^{ε} ;

\neven the slow process Y_t^{ε} has a fast varying component, which is known to be closely related to the homogenization in PDEs.

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- \diamond even the slow process Y_{t}^{ε} has a fast varying component, which is known to be closely related to the homogenization in PDEs.

Main results

Consider the following multiscale SDE in $\mathbb{R}^{d_1+d_2}$:

\n The results\n
$$
\text{index 1: } \int \mathrm{d}x_t^\varepsilon = \alpha_\varepsilon^{-2} b(X_t^\varepsilon, Y_t^\varepsilon) \mathrm{d}t + \beta_\varepsilon^{-1} c(X_t^\varepsilon, Y_t^\varepsilon) \mathrm{d}t + \alpha_\varepsilon^{-1} \sigma(X_t^\varepsilon, Y_t^\varepsilon) \mathrm{d}W_t^1,
$$
\n

\n\n $\begin{cases}\n \mathrm{d}X_t^\varepsilon = \mathcal{C}_\varepsilon^{-2} b(X_t^\varepsilon, Y_t^\varepsilon) \mathrm{d}t + \beta_\varepsilon^{-1} \mathcal{C}(X_t^\varepsilon, Y_t^\varepsilon) \mathrm{d}t + \alpha_\varepsilon^{-1} \sigma(X_t^\varepsilon, Y_t^\varepsilon) \mathrm{d}W_t^1, \\
 \mathrm{d}Y_t^\varepsilon = \mathcal{F}(X_t^\varepsilon, Y_t^\varepsilon) \mathrm{d}t + \gamma_\varepsilon^{-1} \mathcal{H}(X_t^\varepsilon, Y_t^\varepsilon) \mathrm{d}t + \mathcal{G}(Y_t^\varepsilon) \mathrm{d}W_t^2, \\
 \chi_0^\varepsilon = x \in \mathbb{R}^{d_1}, \quad Y_0^\varepsilon = y \in \mathbb{R}^{d_2}, \\
 \mathrm{where}\n \end{cases}$ \n

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$$
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 \diamond there exist two time scales in the fast motion $X_t^\varepsilon;$

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History results:

- Papanicolaou, Stroock and Varadhan (1976);
- Pardoux and Veretennikov (2001, 03, 05, AOP);
- Khasminskii and Yin (2007, JDE).

Main results

Consider the following multiscale SDE in $\mathbb{R}^{d_1+d_2}$:

1.11 **results**

\n1.23 **in** results

\n1.33 **in** results

\n1.44 **in** follows:

\n
$$
\begin{cases}\n\frac{dX_t^{\varepsilon}}{dt} = \alpha_{\varepsilon}^{-2}b(X_t^{\varepsilon}, Y_t^{\varepsilon})dt + \beta_{\varepsilon}^{-1}c(X_t^{\varepsilon}, Y_t^{\varepsilon})dt + \alpha_{\varepsilon}^{-1}\sigma(X_t^{\varepsilon}, Y_t^{\varepsilon})dW_t^1, \\
\frac{dY_t^{\varepsilon}}{dt} = F(X_t^{\varepsilon}, Y_t^{\varepsilon})dt + \gamma_{\varepsilon}^{-1}H(X_t^{\varepsilon}, Y_t^{\varepsilon})dt + G(Y_t^{\varepsilon})dW_t^2, \\
X_0^{\varepsilon} = x \in \mathbb{R}^{d_1}, \quad Y_0^{\varepsilon} = y \in \mathbb{R}^{d_2}, \\
\text{2.33 ar circular case:
$$
\begin{cases}\n\frac{dV_t^{\varepsilon}}{dt} = -\varepsilon^{-1/2}\nabla V(Y_t^{\varepsilon})dt - \varepsilon^{-1}\gamma(Y_t^{\varepsilon})V_t^{\varepsilon}dt + \varepsilon^{-1/2}\sigma(Y_t^{\varepsilon})dW_t^1, \\
\frac{dY_t^{\varepsilon}}{dt} = \varepsilon^{-1/2}V_t^{\varepsilon}dt, \\
\text{3.45 in equivalent to the the overdamped stochastic Langevin equation:
$$
\varepsilon \dot{Y}_t^{\varepsilon} = -\nabla V(Y_t^{\varepsilon}) - \gamma(Y_t^{\varepsilon})Y_t^{\varepsilon} + \sigma(Y_t^{\varepsilon})W_t. \\
\text{4.47 in (1.8781)}\n\end{cases}
$$
\n1.5 **1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1 1.1** <
$$
$$

where the small parameters $\alpha_{\varepsilon}, \beta_{\varepsilon}, \gamma_{\varepsilon} \to 0$ as $\varepsilon \to 0$. (2.3)

A particular case:

$$
\begin{cases} dV_t^{\varepsilon} = -\varepsilon^{-1/2} \nabla \mathcal{V}(\mathbf{Y}_t^{\varepsilon}) dt - \varepsilon^{-1} \gamma(\mathbf{Y}_t^{\varepsilon}) V_t^{\varepsilon} dt + \varepsilon^{-1/2} \sigma(\mathbf{Y}_t^{\varepsilon}) dW_t^1, \\ dY_t^{\varepsilon} = \varepsilon^{-1/2} V_t^{\varepsilon} dt, \end{cases}
$$

which is equivalent to the the overdamped stochastic Langevin equation:

$$
\varepsilon \ddot{Y}_t^\varepsilon = -\nabla \mathcal{V}(Y_t^\varepsilon) - \gamma (Y_t^\varepsilon) Y_t^\varepsilon + \sigma (Y_t^\varepsilon) \dot{W}_t.
$$

Main results - functional LLN

We need to study the following Poisson equation in \mathbb{R}^{d_1} :

$$
\mathscr{L}_0(x,y)\Phi(x,y)=-H(x,y),\quad x\in\mathbb{R}^{d_1},\qquad(2.4)
$$

where $y\in\mathbb{R}^{d_2}$ is a parameter, and $\mathscr{L}_0(x,y)$ is given by

results - functional LLN
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:
\n $\mathcal{L}_0(x, y)\Phi(x, y) = -H(x, y), \quad x \in \mathbb{R}^{d_1},$
\n $\in \mathbb{R}^{d_2}$ is a parameter, and $\mathcal{L}_0(x, y)$ is given by
\n $\mathcal{L}_0(x, y) := \sum_{i,j=1}^{d_1} a^{ij}(x, y) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d_1} b^i(x, y) \frac{\partial}{\partial x_i}$
\n $\phi(x, y) = \sigma(x, y)\sigma^*(x, y).$

with $a(x, y) = \sigma(x, y)\sigma^*(x, y)$.

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$$
\n
$$
y = \sigma(x, y)\sigma^*(x, y).
$$
\nand if H is centered, i.e.,

\n
$$
\int_{\mathbb{R}^{d_1}} H(x, y)\mu^y(\mathrm{d}x) = 0, \quad \forall y \in \mathbb{R}^{d_2}.
$$
\nNotice that $\lim_{x \to \infty} \mathbb{R}^d$ is a solution of the equation of the equation of the equation.

\nSubstituting \mathbb{R}^d and \mathbb{R}^d is a solution.

with $a(x, y) = \sigma(x, y)\sigma^*(x, y)$.

 (A_H) : the drift H is centered, i.e.,

$$
\int_{\mathbb{R}^{d_1}} H(x,y) \mu^y(\mathrm{d} x) = 0, \quad \forall y \in \mathbb{R}^{d_2}.
$$

[Röckner and X., 2020, AOP] $\implies \exists !$ solution Φ to equation [\(2.4\)](#page-23-1).

Depending on the orders that $\alpha_{\varepsilon}, \beta_{\varepsilon}, \gamma_{\varepsilon}$ go to zero, we will have two different regimes of interaction, i.e.,

results - functional LLN
\nng on the orders that
$$
\alpha_{\varepsilon}, \beta_{\varepsilon}, \gamma_{\varepsilon}
$$
 go to zero, we will have two
\nregimes of interaction, i.e.,
\n
$$
\begin{cases}\n\lim_{\varepsilon \to 0} \frac{\alpha_{\varepsilon}}{\gamma_{\varepsilon}} = 0 \text{ and } \lim_{\varepsilon \to 0} \frac{\alpha_{\varepsilon}^2}{\beta_{\varepsilon} \gamma_{\varepsilon}} = 0, \text{ Case 1;} \\
\lim_{\varepsilon \to 0} \frac{\alpha_{\varepsilon}}{\gamma_{\varepsilon}} = 0 \text{ and } \alpha_{\varepsilon}^2 = \beta_{\varepsilon} \gamma_{\varepsilon}, \text{ Case 2.} \n\end{cases}
$$
\n(2.5)
\ne Xie (JSNU)
\n
$$
\text{Multiscale stochastic systems}
$$
\n
$$
\text{Multiscale stochastic systems}
$$
\n
$$
\text{July 15, 2021} \quad 12 \text{ /}
$$

Depending on the orders that $\alpha_{\varepsilon}, \beta_{\varepsilon}, \gamma_{\varepsilon}$ go to zero, we will have two different regimes of interaction, i.e.,

results - functional LLN
\nng on the orders that
$$
\alpha_{\varepsilon}, \beta_{\varepsilon}, \gamma_{\varepsilon}
$$
 go to zero, we will have two
\nregimes of interaction, i.e.,
\n
$$
\begin{cases}\n\lim_{\varepsilon \to 0} \frac{\alpha_{\varepsilon}}{\gamma_{\varepsilon}} = 0 \text{ and } \lim_{\varepsilon \to 0} \frac{\alpha_{\varepsilon}^2}{\beta_{\varepsilon} \gamma_{\varepsilon}} = 0, \text{ Case 1;} \\
\lim_{\varepsilon \to 0} \frac{\alpha_{\varepsilon}}{\gamma_{\varepsilon}} = 0 \text{ and } \alpha_{\varepsilon}^2 = \beta_{\varepsilon} \gamma_{\varepsilon}, \text{ Case 2.} \n\end{cases}
$$
\n(2.5)
\ne Xie (JSNU)
\n
$$
\text{Multiscale stochastic systems}
$$
\n
$$
\text{Multiscale stochastic systems}
$$
\n
$$
\text{July 15, 2021} \quad 12 \text{ /}
$$

Theorem 1 [Röckner and X., (2021) CMP]

The slow process Y^ε_t will converge strongly to $\bar Y^k_t$, where for $k=1,2,$

$$
\mathrm{d}\bar{Y}_t^k = \bar{F}_k(\bar{Y}_t^k)\mathrm{d}t + G(\bar{Y}_t^k)\mathrm{d}W_t^2,
$$

and the averaged drift are given by

1 results - functional LLN

\norem 1 [Röckner and X., (2021) CMP]

\nslow process
$$
Y_t^{\varepsilon}
$$
 will converge strongly to \overline{Y}_t^k , where for $k = 1, 2$,
\n
$$
d\overline{Y}_t^k = \overline{F}_k(\overline{Y}_t^k)dt + G(\overline{Y}_t^k)dW_t^2,
$$
\nthe averaged drift are given by

\n
$$
\overline{F}_1(y) := \int_{\mathbb{R}^{d_1}} F(x, y)\mu^y(dx);
$$
 (Case 1)

\n
$$
\overline{F}_2(y) := \int_{\mathbb{R}^{d_1}} \left[F(x, y) + c(x, y) \cdot \nabla_x \Phi(x, y)\right] \mu^y(dx).
$$
 (Case 2)

\nLongie Xie (JSNU)

\nMultiscale stochastic systems

\blacktriangleright Remark:

When $c=H\equiv 0, \ \alpha_\varepsilon=\sqrt{\varepsilon}$ and $b, F\in C^{\delta,\vartheta}_{x,y},$ we obtain

$$
\sup_{t\in[0,T]}\mathbb{E}|Y_t^{\varepsilon}-\bar{Y}_t^1|\leqslant C_T\,\varepsilon^{(\vartheta\wedge 1)/2}.
$$

example 3
 o = $H \equiv 0$, $\alpha_{\varepsilon} = \sqrt{\varepsilon}$ and $b, F \in C_{x,y}^{\delta,\vartheta}$, we obtain
 $\sup_{t \in [0,T]} \mathbb{E}|Y_t^{\varepsilon} - \bar{Y}_t^1| \leqslant C_T \, \varepsilon^{(\vartheta \wedge 1)/2}.$

that the convergence rate does not dependent on the regular

coefficients w Note that the convergence rate does not dependent on the regularity of the coefficients w.r.t. the fast variable.

Main results - functional CLT

We first study SDE (2.3) with $H \equiv 0$, i.e., there is no fast term in the slow component:

[D](#page-0-0)raft (dX ε ^t = α −2 ^ε b(X ε t , Y ε t)dt + β −1 ε c(X ε t , Y ε t)dt + α −1 ^ε σ(X ε t , Y ε t)dW ¹ t , dY ε ^t = F(X ε t , Y ε t)dt + G(Y ε t)dW ² t .

 QQ

We first study SDE (2.3) with $H \equiv 0$, i.e., there is no fast term in the slow component:

[D](#page-0-0)raft ε −2 ε ε −1 ε ε −1 ε ε)dW ¹ (dX ^t = α ^ε b(X , Y)dt + β c(X , Y)dt + α ^ε σ(X , Y , t t ε t t t t t ε ε ε ε)dW ² dY ^t = F(X , Y)dt + G(Y . t t t t

According to Theorem 1 (Case 1), we have

$$
\mathbb{E}|Y_t^{\varepsilon}-\bar{Y}_t^1|\leqslant C_0(\alpha_{\varepsilon}+\alpha_{\varepsilon}^2/\beta_{\varepsilon}).
$$

We intend to characterize the asymptotic behavior of the normalized difference

$$
Z_t^\varepsilon:=\frac{Y_t^\varepsilon-\bar{Y}_t^1}{\eta_\varepsilon}
$$

with proper deviation scale η_{ε} such that $\eta_{\varepsilon} \to 0$ as $\varepsilon \to 0$.

The natural choice of the deviation scale η_{ε} should be divided into the following three cases:

results - functional CLT
\n1.ural choice of the deviation scale
$$
\eta_{\varepsilon}
$$
 should be divided into t
\ng three cases:
\n
$$
\begin{cases}\n\eta_{\varepsilon} = \frac{\alpha_{\varepsilon}^{2}}{\beta_{\varepsilon}} & \text{and} \quad \lim_{\varepsilon \to 0} \frac{\beta_{\varepsilon}}{\alpha_{\varepsilon}} = 0, & \text{Case 0.1}; \\
\eta_{\varepsilon} = \alpha_{\varepsilon} & \text{and} \quad \lim_{\varepsilon \to 0} \frac{\alpha_{\varepsilon}}{\beta_{\varepsilon}} = 0, & \text{Case 0.2}; \\
\eta_{\varepsilon} = \alpha_{\varepsilon} = \beta_{\varepsilon}, & \text{Case 0.3}. \n\end{cases}
$$

 QQ

Let $\Gamma(x, y)$ be the unique solution of the following Poisson equation:

$$
\mathscr{L}_0(x,y)\Gamma(x,y)=-\big[F(x,y)-\overline{F}_1(y)\big]:=-\widetilde{F}(x,y),
$$

Define

results - functional CLT

\ny) be the unique solution of the following Poisson equation

\n
$$
\mathcal{L}_0(x, y)\Gamma(x, y) = -\left[F(x, y) - \bar{F}_1(y)\right] := -\tilde{F}(x, y),
$$
\n
$$
\overline{c \cdot \nabla_x \Gamma}(y) := \int_{\mathbb{R}^{d_1}} c(x, y) \cdot \nabla_x \Gamma(x, y) \mu^y(\mathrm{d}x),
$$
\n
$$
\overline{\tilde{F} \cdot \Gamma^*}(y) := \int_{\mathbb{R}^{d_1}} \tilde{F}(x, y) \cdot \Gamma^*(x, y) \mu^y(\mathrm{d}x).
$$
\nand

\nand

\nand

\nand

\nand

\n
$$
\overline{f} \cdot \Gamma^*(y) := \int_{\mathbb{R}^{d_1}} \tilde{F}(x, y) \cdot \Gamma^*(x, y) \mu^y(\mathrm{d}x).
$$
\nand

\nand

\n
$$
\overline{f} \cdot \Gamma^*(y) = \int_{\text{fullity 15, 202}} \tilde{F}(x, y) \mu^y(\mathrm{d}x).
$$

э

 QQ

Theorem 2 (CLT: Case 0) [Röckner and X., 2021, CMP]

The limit processes $\bar{\mathsf{Z}}_{\ell,t}^{0}\,\,(\ell=1,2,3)$ for $\mathsf{Z}_{t}^{\varepsilon}$ corresponding to Case 0.1-Case 0.3 satisfy

[D](#page-0-0)raft dZ¯ ⁰ ¹,^t ⁼ [∇]yF¯ ¹(Y¯ ¹ t)Z¯ ⁰ ¹,td^t ⁺ [∇]yG(Y¯ ¹ t)Z¯ ⁰ ¹,tdW ² ^t ⁺ ^c · ∇xΓ(Y¯ ¹ t)dt; dZ¯ ⁰ ²,^t ⁼ [∇]yF¯ ¹(Y¯ ¹ t)Z¯ ⁰ ²,td^t ⁺ [∇]yG(Y¯ ¹ t)Z¯ ⁰ ²,tdW ² ^t + q F˜ · Γ [∗](Y¯ ¹ t)dW˜ t ; dZ¯ ⁰ ³,^t ⁼ [∇]yF¯ ¹(Y¯ ¹ t)Z¯ ⁰ ³,td^t ⁺ [∇]yG(Y¯ ¹ t)Z¯ ⁰ ³,tdW ² t + c · ∇xΓ(Y¯ ¹ t)dt + q F˜ · Γ [∗](Y¯ ¹ t)dW˜ t ,

where \tilde{W}_t is another Brownian motion independent of $W_t^2.$

\blacktriangleright Remark:

Ound - functional CLT

icular, when $c = H \equiv 0$, $\alpha_{\varepsilon} = \sqrt{\varepsilon}$ and $b, F \in C_{x,y}^{\delta,1+\vartheta}$ with
 $0,$ we have
 $\sup_{t \in [0,T]} \Big| \mathbb{E}[\varphi(Z_t^\varepsilon)] - \mathbb{E}[\varphi(\bar{Z}_t)] \Big| \leqslant C_T \, \varepsilon^{(\vartheta \wedge 1)/2},$
 $\overline{Z}_t = \nabla_y \bar{F}(\bar{Y}_t) \bar{Z}_$ In particular, when $c = H \equiv 0$, $\alpha_{\varepsilon} = \sqrt{\varepsilon}$ and $b, F \in C_{x,y}^{\delta,1+\vartheta}$ with $\delta, \vartheta > 0$, we have

$$
\sup_{t\in[0,T]}\left|\mathbb{E}[\varphi(Z_t^{\varepsilon})] - \mathbb{E}[\varphi(\bar{Z}_t)]\right| \leqslant C_T \, \varepsilon^{(\vartheta\wedge 1)/2},
$$

where

$$
\mathrm{d}\bar{Z}_t=\nabla_y\bar{F}(\bar{Y}_t)\bar{Z}_t\mathrm{d}t+\nabla_yG(\bar{Y}^1_t)\bar{Z}_t\mathrm{d}W_t^2+\sqrt{\overline{\tilde{F}}\cdot\Gamma^*}(\bar{Y}_t)\mathrm{d}\tilde{W}_t.
$$

Main results - functional CLT

esults – functional CLT

consider Case 1 in (2.5). According to Theorem 1 again, w
 $\mathbb{E}|Y_t^\varepsilon - \bar{Y}_t^1| \leqslant C_1(\alpha_\varepsilon/\gamma_\varepsilon + \alpha_\varepsilon^2/(\beta_\varepsilon \gamma_\varepsilon)).$

ne normalized difference
 $Z_t^{1,\varepsilon} := \frac{Y_t^\varepsilon - \bar{Y}_t^1}{\eta_\varepsilon}.$
 $\$ Now, we consider Case 1 in (2.5). According to Theorem 1 again, we have

$$
\mathbb{E}|Y_t^{\varepsilon}-\bar{Y}_t^1|\leqslant C_1(\alpha_{\varepsilon}/\gamma_{\varepsilon}+\alpha_{\varepsilon}^2/(\beta_{\varepsilon}\gamma_{\varepsilon}))\,.
$$

Define the normalized difference

$$
Z_t^{1,\varepsilon}:=\frac{Y_t^\varepsilon-\bar{Y}_t^1}{\eta_\varepsilon}.
$$

Main results - functional CLT

Now, we consider Case 1 in (2.5). According to Theorem 1 again, we have

$$
\mathbb{E}|Y_t^{\varepsilon}-\bar{Y}_t^1|\leqslant C_1(\alpha_{\varepsilon}/\gamma_{\varepsilon}+\alpha_{\varepsilon}^2/(\beta_{\varepsilon}\gamma_{\varepsilon}))\,.
$$

Define the normalized difference

$$
Z_t^{1,\varepsilon}:=\frac{Y_t^\varepsilon-\bar{Y}_t^1}{\eta_\varepsilon}.
$$

Then the natural choice of the deviation scale η_{ε} in order to observe non-trivial behavior for $Z_t^{1,\varepsilon}$ should be divided into the following three cases:

results - functional CLT

\nconsider Case 1 in (2.5). According to Theorem 1 again, we have

\n
$$
\mathbb{E}|Y_t^{\varepsilon} - \bar{Y}_t^1| \leq C_1(\alpha_{\varepsilon}/\gamma_{\varepsilon} + \alpha_{\varepsilon}^2/(\beta_{\varepsilon}\gamma_{\varepsilon}))
$$
\nne normalized difference

\n
$$
Z_t^{1,\varepsilon} := \frac{Y_t^{\varepsilon} - \bar{Y}_t^1}{\eta_{\varepsilon}}
$$
\ne natural choice of the deviation scale

\n
$$
\eta_{\varepsilon}
$$
\nin order to observe

\nal behavior for

\n
$$
Z_t^{1,\varepsilon}
$$
\nshould be divided into the following three

\n
$$
\eta_{\varepsilon} = \frac{\alpha_{\varepsilon}^2}{\beta_{\varepsilon}\gamma_{\varepsilon}}
$$
\nand

\n
$$
\lim_{\varepsilon \to 0} \frac{\beta_{\varepsilon}}{\alpha_{\varepsilon}} = 0,
$$
\nCase 1.1;

\n
$$
\eta_{\varepsilon} = \frac{\alpha_{\varepsilon}}{\gamma_{\varepsilon}}
$$
\nand

\n
$$
\lim_{\varepsilon \to 0} \frac{\alpha_{\varepsilon}}{\beta_{\varepsilon}} = 0,
$$
\nCase 1.2;

\n
$$
\eta_{\varepsilon} = \frac{\alpha_{\varepsilon}}{\gamma_{\varepsilon}}
$$
\nand

\n
$$
\alpha_{\varepsilon} = \beta_{\varepsilon},
$$
\nCase 1.3.

\nAsive (JSNU)

\nMultiscale stochastic systems

Recall that Φ is the unique solution to the Poisson equation

$$
\mathscr{L}_0(x,y)\Phi(x,y)=-H(x,y).
$$

Define

results - functional CLT

\nthat
$$
\Phi
$$
 is the unique solution to the Poisson equation

\n
$$
\mathcal{L}_0(x, y)\Phi(x, y) = -H(x, y).
$$
\n
$$
\overline{c \cdot \nabla_x \Phi}(y) := \int_{\mathbb{R}^{d_1}} c(x, y) \cdot \nabla_x \Phi(x, y) \mu^y(\mathrm{d}x);
$$
\n
$$
\overline{H \cdot \Phi^*}(y) := \int_{\mathbb{R}^{d_1}} H(x, y) \cdot \Phi^*(x, y) \mu^y(\mathrm{d}x).
$$
\nand

\nand

\nand

\n
$$
\overline{C} \cdot \overline{C
$$

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Theorem 3 (CLT: Case 1) [Röckner and X., 2021, CMP]

The limiting processes $\bar{\mathcal{Z}}_{\ell,t}^1$ $(\ell=1,2,3)$ for $\mathcal{Z}_t^{1,\varepsilon}$ corresponding to Case 1.1-Case 1.3 satisfy

18.1.
$$
f(x) = \frac{1}{2} \int_{\ell}^{2} \left[\frac{1}{2} \int_{\ell}^{2} f(x) \right]
$$

\n19.1. $f(x) = \frac{1}{2} \int_{\ell}^{2} f(x) \, dx$

\n10.1. $f(x) = \frac{1}{2} \int_{\ell}^{2} f(x) \, dx$

\n11. $f(x) = \frac{1}{2} \int_{\ell}^{2} f(x) \, dx$

\n12. $f(x) = \frac{1}{2} \int_{\ell}^{2} f(x) \, dx$

\n13. $f(x) = \frac{1}{2} \int_{\ell}^{2} f(x) \, dx$

\n14. $f(x) = \frac{1}{2} \int_{\ell}^{2} f(x) \, dx$

\n15. $f(x) = \frac{1}{2} \int_{\ell}^{2} f(x) \, dx$

\n16. $f(x) = \frac{1}{2} \int_{\ell}^{2} f(x) \, dx$

\n17. $f(x) = \frac{1}{2} \int_{\ell}^{2} f(x) \, dx$

\n18. $f(x) = \frac{1}{2} \int_{\ell}^{2} f(x) \, dx$

\n19. $f(x) = \frac{1}{2} \int_{\ell}^{2} f(x) \, dx$

\n10. $f(x) = \frac{1}{2} \int_{\ell}^{2} f(x) \, dx$

\n11. $f(x) = \frac{1}{2} \int_{\ell}^{2} f(x) \, dx$

\n12. <math display="inline</p>

where \tilde{W}_t is another Brownian motion independent of $W_t^2.$

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Main results - functional CLT

esults – functional CLT

we consider Case 2 in (2.5), where homogenization already (

the LLN. According to Theorem 1 (Case 2), we have
 $\mathbb{E}|Y_t^{\varepsilon} - \bar{Y}_t^2| \leqslant C_2(\alpha_{\varepsilon}/\gamma_{\varepsilon} + \alpha_{\varepsilon}^2/\beta_{\varepsilon}).$

ne normalized Finally, we consider Case 2 in (2.5), where homogenization already occurs even in the LLN. According to Theorem 1 (Case 2), we have

$$
\mathbb{E}|Y_t^{\varepsilon}-\bar{Y}_t^2|\leqslant C_2(\alpha_{\varepsilon}/\gamma_{\varepsilon}+\alpha_{\varepsilon}^2/\beta_{\varepsilon}).
$$

Define the normalized difference

$$
Z_t^{2,\varepsilon}:=\frac{Y_t^\varepsilon-\bar{Y}_t^2}{\eta_\varepsilon}
$$

.

Main results - functional CLT

Finally, we consider Case 2 in (2.5), where homogenization already occurs even in the LLN. According to Theorem 1 (Case 2), we have

$$
\mathbb{E}|Y_t^{\varepsilon}-\bar{Y}_t^2|\leqslant C_2(\alpha_{\varepsilon}/\gamma_{\varepsilon}+\alpha_{\varepsilon}^2/\beta_{\varepsilon}).
$$

Define the normalized difference

 \sim

$$
Z_t^{2,\varepsilon}:=\frac{Y_t^{\varepsilon}-\bar{Y}_t^2}{\eta_{\varepsilon}}
$$

The natural choice of the derivation scale η_{ε} should be divided into the following three cases:

results - functional CLT

\nwe consider Case 2 in (2.5), where homogenization already occurs, the LLN. According to Theorem 1 (Case 2), we have

\n
$$
\mathbb{E}|Y_t^{\varepsilon} - \bar{Y}_t^2| \leq C_2(\alpha_{\varepsilon}/\gamma_{\varepsilon} + \alpha_{\varepsilon}^2/\beta_{\varepsilon}).
$$
\nne normalized difference

\n
$$
Z_t^{2,\varepsilon} := \frac{Y_t^{\varepsilon} - \bar{Y}_t^2}{\eta_{\varepsilon}}.
$$
\nural choice of the derivation scale

\n
$$
\eta_{\varepsilon}
$$
\nwhere cases:

\n
$$
\eta_{\varepsilon} = \frac{\alpha_{\varepsilon}^2}{\beta_{\varepsilon}}
$$
\nand

\n
$$
\lim_{\varepsilon \to 0} \frac{\beta_{\varepsilon}}{\alpha_{\varepsilon} \gamma_{\varepsilon}} = 0,
$$
\nCase 2.1;

\n
$$
\eta_{\varepsilon} = \frac{\alpha_{\varepsilon}}{\gamma_{\varepsilon}}
$$
\nand

\n
$$
\lim_{\varepsilon \to 0} \frac{\alpha_{\varepsilon} \gamma_{\varepsilon}}{\beta_{\varepsilon}} = 0,
$$
\nCase 2.2;

\n
$$
\eta_{\varepsilon} = \frac{\alpha_{\varepsilon}}{\gamma_{\varepsilon}} = \frac{\alpha_{\varepsilon}^2}{\beta_{\varepsilon}},
$$
\nCase 2.3.

\nAs:

\nExercise (15NU)

\nMultiscale stochastic systems

.

Recall that

$$
\mathscr{L}_0(x,y)\Phi(x,y)=-H(x,y),\ \mathscr{L}_0(x,y)\Gamma(x,y)=-\big[F(x,y)-\overline{F}_1(y)\big].
$$

Let Ψ solves the following Poisson equation:

$$
\mathscr{L}_0(x,y)\Psi(x,y)=-\big[c(x,y)\cdot\nabla_x\Phi(x,y)-\overline{c\cdot\nabla_x\Phi}(y)\big].
$$

Denote by

results – functional CLT

\nnat

\n
$$
y)\Phi(x,y) = -H(x,y), \ \mathcal{L}_0(x,y)\Gamma(x,y) = -[F(x,y) - \bar{F}_1(y)]
$$
\nolves the following Poisson equation:

\n
$$
\mathcal{L}_0(x,y)\Psi(x,y) = -[c(x,y)\cdot\nabla_x\Phi(x,y) - \overline{c}\cdot\nabla_x\Phi(y)].
$$
\nby

\n
$$
\overline{c\cdot\nabla_x\Psi}(y) := \int_{\mathbb{R}^{d_1}} c(x,y) \cdot \nabla_x\Psi(x,y)\mu^y(\mathrm{d}x).
$$
\nMultiple stochastic systems

 QQ

Theorem 4 (CLT: Case 2) [Röckner and X., 2021, CMP]

The limiting processes $\bar{Z}_{\ell,t}^2$ $(\ell=1,2,3)$ for $Z_t^{2,\varepsilon}$ corresponding to Case 2.1-Case 2.3 satisfy

lain results - functional CLT
\ntheorem 4 (CLT: Case 2) [Röckner and X., 2021, CMP]
\nhe limiting processes
$$
\bar{Z}_{\ell,t}^2
$$
 ($\ell = 1, 2, 3$) for $Z_t^{2,\varepsilon}$ corresponding to
\nase 2.1-Case 2.3 satisfy
\n
$$
d\bar{Z}_{1,t}^2 = \nabla_y \bar{F}_2(\bar{Y}_t^2) \bar{Z}_{1,t}^2 dt + \nabla_y G(\bar{Y}_t^2) \bar{Z}_{1,t}^2 dW_t^2 + (\bar{C} \cdot \nabla_x \bar{V}_t + \bar{C} \cdot \nabla_x \bar{V}_t)(\bar{Y}_t^2) dt;
$$
\n
$$
d\bar{Z}_{2,t}^2 = \nabla_y \bar{F}_2(\bar{Y}_t^2) \bar{Z}_{2,t}^2 dt + \nabla_y G(\bar{Y}_t^2) \bar{Z}_{2,t}^2 dW_t^2 + \sqrt{H \cdot \Phi^*}(\bar{Y}_t^2) d\tilde{W}_t;
$$
\n
$$
d\bar{Z}_{3,t}^2 = \nabla_y \bar{F}_2(\bar{Y}_t^2) \bar{Z}_{3,t}^2 dt + \nabla_y G(\bar{Y}_t^2) \bar{Z}_{3,t}^2 dW_t^2 + (\bar{C} \cdot \nabla_x \bar{V}_t + \bar{C} \cdot \nabla_x \bar{V}_t)(\bar{Y}_t^2) dt + \sqrt{H \cdot \Phi^*}(\bar{Y}_t^2) d\tilde{W}_t,
$$
\nthere \tilde{W}_t is another Brownian motion independent of W_t^2 .
\nLongie Xie (JSNU)
\nMultiscale stochastic systems

where \tilde{W}_t is another Brownian motion independent of $W_t^2.$

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EXE (JSNU)

Multiscale stochastic systems

Multiscale stochastic systems

Multiscale stochastic systems
 [D](#page-0-0)uly 15, 202 Thank You !

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