Limit theorems for multiscale stochastic dynamical systems

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Overview

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   - Normal deviations - functional CLT

2. Main results
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   - Functional CLT: Case 0
   - Functional CLT: Case 1
   - Functional CLT: Case 2

Background - Averaging principle

Consider the two-time-scales stochastic system:

$$\mathrm{d}Y_t^\varepsilon = F(X_{t/\varepsilon}, Y_t^\varepsilon)\mathrm{d}t + \mathrm{d}W_t, \quad Y_0 = y \in \mathbb{R}^d \tag{1.1}$$

where $X = (X_t)_{t\geqslant 0}$ is an ergodic Markov process possessing a unique invariant measure $\mu(\mathrm{d}x)$, and $0 < \varepsilon \ll 1$ represents the separation of time scales.

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▷ Y_t^ε (slow variable): mathematical model for a phenomenon appearing at the natural time scale;

▷ $X_{t/\varepsilon}$ (fast variable): fast random environment/effects at a faster time scale (with time order $1/\varepsilon$).

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Usually, the system (1.1) is difficult to deal with due to the two widely separated time scales. Thus a simplified equation which governs the evolution of the system over a long time scale (as $\varepsilon \to 0$) is highly desirable.

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Intuitively,

$$X_{t/\varepsilon} \Rightarrow \mu(\mathrm{d}x) \quad \text{as} \quad \varepsilon \to 0.$$

Thus, by averaging the coefficient with respect to the fast variable, the slow part Y_t^ε will converge to $\bar{Y}_t$, where

$$\mathrm{d}\bar{Y}_t = \bar{F}(\bar{Y}_t)\mathrm{d}t + \mathrm{d}W_t, \quad Y_0 = y \in \mathbb{R}^d$$

with

$$\bar{F}(y) := \int_{\mathbb{R}^d} F(x, y)\mu(\mathrm{d}x).$$

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This theory, known as the averaging principle, was first developed by Bogolyubov (1937) for ODEs and extended to the SDEs by Khasminskii (1966).

Background - Averaging principle

Consider the following fast-slow stochastic system in $\mathbb{R}^{d_1+d_2}$:

$$\begin{cases} \mathrm{d}X_t^\varepsilon = \varepsilon^{-1} b(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + \varepsilon^{-1/2}\sigma(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}W_t^1, \\ \mathrm{d}Y_t^\varepsilon = F(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + G(Y_t^\varepsilon)\mathrm{d}W_t^2, \\ X_0^\varepsilon = x \in \mathbb{R}^{d_1}, \quad Y_0^\varepsilon = y \in \mathbb{R}^{d_2}, \end{cases} \tag{1.2}$$

where $0 < \varepsilon \ll 1$ is a small parameter.

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where $0 < \varepsilon \ll 1$ is a small parameter.

The intuitive idea for deriving a simplified equation for (1.2) is based on the observation that:

- ◇ during the fast transients, the slow variable remains "constant";
- ◇ by the time its changes become noticeable, the fast variable has almost reached its quasi-steady state.

Background - Averaging principle

$\diamond$ With the natural time scaling $t \mapsto \varepsilon t$, the process $\tilde{X}_t^\varepsilon := X_{\varepsilon t}^\varepsilon$ satisfies

$$\mathrm{d}\tilde{X}_t^\varepsilon = b(\tilde{X}_t^\varepsilon, Y_{\varepsilon t}^\varepsilon)\mathrm{d}t + \sigma(\tilde{X}_t^\varepsilon, Y_{\varepsilon t}^\varepsilon)\mathrm{d}\tilde{W}_t^1,$$

where $\tilde{W}_t^1 := \varepsilon^{-1/2} W_{\varepsilon t}^1$ is a new BM.

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Thus we need to consider the auxiliary process X_t^y which satisfies the frozen equation

$$\mathrm{d}X_t^y = b(X_t^y, y)\mathrm{d}t + \sigma(X_t^y, y)\mathrm{d}W_t^1, \quad X_0^y = x \in \mathbb{R}^{d_1}.$$

Under certain recurrence conditions, the process X_t^y process a unique invariant measure $\mu^y(\mathrm{d}x)$.

Background - Averaging principle

$\diamond$ Then by averaging the coefficients with respect to parameter in fast variable, the slow part Y_t^ε will converge to $\bar{Y}_t$, where

$$\mathrm{d}\bar{Y}_t = \bar{F}(\bar{Y}_t)\mathrm{d}t + G(\bar{Y}_t)\mathrm{d}W_t^2$$

with

$$\bar{F}(y) := \int_{\mathbb{R}^{d_1}} F(x, y)\mu^y(\mathrm{d}x).$$

Background - Averaging principle

▷ Strong convergence:

$$\sup_{t\in[0,T]} \mathbb{E}|Y_t^\varepsilon - \bar{Y}_t| \leqslant C_T\,\varepsilon^{1/2}.$$

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History results:

- Freidlin and Wentcell (1998), Pavliotis and Stuart (2008), $\cdots$.
  Condition: all the coefficients are Lipschitz continuous.

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- If G depends on the fast variable x, then the strong convergence may not hold.

- The convergence rate is important for numerical schemes (called HMM) for multiscale systems (see e.g. [E. etc, 2005, CPAM]).

Background - Normal deviation

However, the time scale separation is never infinite in reality.

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For small but positive ε, the process Y_t^ε will experience fluctuations around its average $\bar{Y}_t$.

To leading order, these fluctuations can be captured by characterizing the asymptotic behavior of the normalized difference

$$Z_t^\varepsilon := \frac{Y_t^\varepsilon - \bar{Y}_t}{\sqrt{\varepsilon}} = \frac{1}{\sqrt{\varepsilon}} \int_0^t \big[F(X_s^\varepsilon, Y_s^\varepsilon) - \bar{F}(\bar{Y}_s)\big] \mathrm{d}s$$

as ε tends to 0.

Background - Normal deviation

When $G \equiv \mathbb{I}_{d_2}$, the deviation process Z_t^ε is known to converge weakly to $\bar{Z}_t$ with

$$\mathrm{d}\bar{Z}_t = \nabla_y \bar{F}(\bar{Y}_t)\bar{Z}_t \mathrm{d}t + \zeta(\bar{Y}_t)\mathrm{d}\tilde{W}_t,$$

where $\tilde{W}_t$ is another standard Brownian motion, and the new diffusion coefficient is given by

$$\zeta(y) := \sqrt{\int_0^\infty \int_{\mathbb{R}^{d_1}} \mathbb{E}\big[F(X_t^y(x), y) - \bar{F}(y)\big]\big[F(x, y) - \bar{F}(y)\big]^* \mu^y(\mathrm{d}x)\mathrm{d}t}.$$

Such result, also known as the Gaussian approximation, is an analogue of the functional central limit theorem.

Consider the following multiscale SDE in $\mathbb{R}^{d_1+d_2}$:

$$\begin{cases} \mathrm{d}X_t^\varepsilon = \alpha_\varepsilon^{-2} b(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + \beta_\varepsilon^{-1} c(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + \alpha_\varepsilon^{-1}\sigma(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}W_t^1, \\ \mathrm{d}Y_t^\varepsilon = F(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + \gamma_\varepsilon^{-1} H(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + G(Y_t^\varepsilon)\mathrm{d}W_t^2, \\ X_0^\varepsilon = x \in \mathbb{R}^{d_1}, \quad Y_0^\varepsilon = y \in \mathbb{R}^{d_2}, \end{cases} \tag{2.3}$$

where the small parameters $\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon \to 0$ as $\varepsilon \to 0$.

Main results

Consider the following multiscale SDE in $\mathbb{R}^{d_1+d_2}$:

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- ◇ there exist two time scales in the fast motion X_t^ε;
- ◇ even the slow process Y_t^ε has a fast varying component, which is known to be closely related to the homogenization in PDEs.

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- ◇ even the slow process Y_t^ε has a fast varying component, which is known to be closely related to the homogenization in PDEs.

History results:

- Papanicolaou, Stroock and Varadhan (1976);
- Pardoux and Veretennikov (2001, 03, 05, AOP);
- Khasminskii and Yin (2007, JDE).

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where the small parameters $\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon \to 0$ as $\varepsilon \to 0$.

A particular case:

$$\begin{cases} \mathrm{d}V_t^\varepsilon = -\varepsilon^{-1/2}\nabla\mathcal{V}(Y_t^\varepsilon)\mathrm{d}t - \varepsilon^{-1}\gamma(Y_t^\varepsilon)V_t^\varepsilon\mathrm{d}t + \varepsilon^{-1/2}\sigma(Y_t^\varepsilon)\mathrm{d}W_t^1, \\ \mathrm{d}Y_t^\varepsilon = \varepsilon^{-1/2}V_t^\varepsilon\mathrm{d}t, \end{cases}$$

which is equivalent to the the overdamped stochastic Langevin equation:

$$\varepsilon\ddot{Y}_t^\varepsilon = -\nabla\mathcal{V}(Y_t^\varepsilon) - \gamma(Y_t^\varepsilon)\dot{Y}_t^\varepsilon + \sigma(Y_t^\varepsilon)\dot{W}_t.$$

Main results - functional LLN

We need to study the following Poisson equation in $\mathbb{R}^{d_1}$:

$$\mathscr{L}_0(x,y)\Phi(x,y) = -H(x,y), \quad x \in \mathbb{R}^{d_1}, \tag{2.4}$$

where $y \in \mathbb{R}^{d_2}$ is a parameter, and $\mathscr{L}_0(x,y)$ is given by

$$\mathscr{L}_0(x,y) := \sum_{i,j=1}^{d_1} a^{ij}(x,y)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d_1} b^i(x,y)\frac{\partial}{\partial x_i}$$

with $a(x,y) = \sigma(x,y)\sigma^*(x,y)$.

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with $a(x,y) = \sigma(x,y)\sigma^*(x,y)$.

($\mathbf{A}_H$): the drift H is centered, i.e.,

$$\int_{\mathbb{R}^{d_1}} H(x,y)\mu^y(\mathrm{d}x) = 0, \quad \forall y \in \mathbb{R}^{d_2}.$$

[Röckner and X., 2020, AOP] $\Longrightarrow \exists$! solution Φ to equation (2.4).

Main results - functional LLN

Depending on the orders that $\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon$ go to zero, we will have two different regimes of interaction, i.e.,

$$\begin{cases} \lim_{\varepsilon\to 0} \frac{\alpha_\varepsilon}{\gamma_\varepsilon} = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \frac{\alpha_\varepsilon^2}{\beta_\varepsilon\gamma_\varepsilon} = 0, \quad \text{Case 1}; \\ \lim_{\varepsilon\to 0} \frac{\alpha_\varepsilon}{\gamma_\varepsilon} = 0 \quad \text{and} \quad \alpha_\varepsilon^2 = \beta_\varepsilon\gamma_\varepsilon, \quad \text{Case 2}. \end{cases} \tag{2.5}$$

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Main results - functional LLN

Theorem 1 [Röckner and X., (2021) CMP]

The slow process Y_t^ε will converge strongly to $\bar{Y}_t^k$, where for $k = 1, 2$,

$$\mathrm{d}\bar{Y}_t^k = \bar{F}_k(\bar{Y}_t^k)\mathrm{d}t + G(\bar{Y}_t^k)\mathrm{d}W_t^2,$$

and the averaged drift are given by

$$\bar{F}_1(y) := \int_{\mathbb{R}^{d_1}} F(x, y)\mu^y(\mathrm{d}x); \qquad \text{(Case 1)}$$

$$\bar{F}_2(y) := \int_{\mathbb{R}^{d_1}} \big[F(x, y) + c(x, y) \cdot \nabla_x \Phi(x, y)\big]\mu^y(\mathrm{d}x). \qquad \text{(Case 2)}$$

- **Remark:**

  When $c = H \equiv 0$, $\alpha_\varepsilon = \sqrt{\varepsilon}$ and $b, F \in C_{x,y}^{\delta,\vartheta}$, we obtain

$$\sup_{t\in[0,T]} \mathbb{E}|Y_t^\varepsilon - \bar{Y}_t^1| \leqslant C_T\, \varepsilon^{(\vartheta\wedge 1)/2}.$$

  Note that the convergence rate does not dependent on the regularity of the coefficients w.r.t. the fast variable.

Main results - functional CLT

We first study SDE (2.3) with $H \equiv 0$, i.e., there is no fast term in the slow component:

$$\begin{cases} \mathrm{d}X_t^\varepsilon = \alpha_\varepsilon^{-2} b(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + \beta_\varepsilon^{-1} c(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + \alpha_\varepsilon^{-1}\sigma(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}W_t^1, \\ \mathrm{d}Y_t^\varepsilon = F(X_t^\varepsilon, Y_t^\varepsilon)\mathrm{d}t + G(Y_t^\varepsilon)\mathrm{d}W_t^2. \end{cases}$$

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According to Theorem 1 (Case 1), we have

$$\mathbb{E}|Y_t^\varepsilon - \bar{Y}_t^1| \leqslant C_0\big(\alpha_\varepsilon + \alpha_\varepsilon^2/\beta_\varepsilon\big).$$

We intend to characterize the asymptotic behavior of the normalized difference

$$Z_t^\varepsilon := \frac{Y_t^\varepsilon - \bar{Y}_t^1}{\eta_\varepsilon}$$

with proper deviation scale η_ε such that $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$.

The natural choice of the deviation scale η_ε should be divided into the following three cases:

$$\begin{cases} \eta_\varepsilon = \dfrac{\alpha_\varepsilon^2}{\beta_\varepsilon} \quad \text{and} \quad \lim\limits_{\varepsilon\to 0} \dfrac{\beta_\varepsilon}{\alpha_\varepsilon} = 0, & \text{Case 0.1}; \\ \eta_\varepsilon = \alpha_\varepsilon \quad \text{and} \quad \lim\limits_{\varepsilon\to 0} \dfrac{\alpha_\varepsilon}{\beta_\varepsilon} = 0, & \text{Case 0.2}; \\ \eta_\varepsilon = \alpha_\varepsilon = \beta_\varepsilon, & \text{Case 0.3}. \end{cases}$$

Main results - functional CLT

Let $\Gamma(x, y)$ be the unique solution of the following Poisson equation:

$$\mathscr{L}_0(x, y)\Gamma(x, y) = -\big[F(x, y) - \bar{F}_1(y)\big] := -\tilde{F}(x, y),$$

Define

$$\overline{c \cdot \nabla_x \Gamma}(y) := \int_{\mathbb{R}^{d_1}} c(x, y) \cdot \nabla_x \Gamma(x, y) \mu^y(\mathrm{d}x),$$
$$\overline{\tilde{F} \cdot \Gamma^*}(y) := \int_{\mathbb{R}^{d_1}} \tilde{F}(x, y) \cdot \Gamma^*(x, y) \mu^y(\mathrm{d}x).$$

Main results - functional CLT

Theorem 2 (CLT: Case 0) [Röckner and X., 2021, CMP]

The limit processes $\bar{Z}^0_{\ell,t}$ $(\ell = 1, 2, 3)$ for Z^ε_t corresponding to Case 0.1-Case 0.3 satisfy

$$\mathrm{d}\bar{Z}^0_{1,t} = \nabla_y \bar{F}_1(\bar{Y}^1_t)\bar{Z}^0_{1,t}\mathrm{d}t + \nabla_y G(\bar{Y}^1_t)\bar{Z}^0_{1,t}\mathrm{d}W^2_t + \overline{c\cdot\nabla_x\Gamma}(\bar{Y}^1_t)\mathrm{d}t;$$

$$\mathrm{d}\bar{Z}^0_{2,t} = \nabla_y \bar{F}_1(\bar{Y}^1_t)\bar{Z}^0_{2,t}\mathrm{d}t + \nabla_y G(\bar{Y}^1_t)\bar{Z}^0_{2,t}\mathrm{d}W^2_t + \sqrt{\overline{\tilde{F}\cdot\Gamma^*}(\bar{Y}^1_t)}\mathrm{d}\tilde{W}_t;$$

$$\begin{aligned}\mathrm{d}\bar{Z}^0_{3,t} &= \nabla_y \bar{F}_1(\bar{Y}^1_t)\bar{Z}^0_{3,t}\mathrm{d}t + \nabla_y G(\bar{Y}^1_t)\bar{Z}^0_{3,t}\mathrm{d}W^2_t \\ &\quad + \overline{c\cdot\nabla_x\Gamma}(\bar{Y}^1_t)\mathrm{d}t + \sqrt{\overline{\tilde{F}\cdot\Gamma^*}(\bar{Y}^1_t)}\mathrm{d}\tilde{W}_t,\end{aligned}$$

where $\tilde{W}_t$ is another Brownian motion independent of W^2_t.

- **Remark:**
  In particular, when $c = H \equiv 0$, $\alpha_\varepsilon = \sqrt{\varepsilon}$ and $b, F \in C^{\delta, 1+\vartheta}_{x,y}$ with $\delta, \vartheta > 0$, we have

$$\sup_{t\in[0,T]} \left| \mathbb{E}[\varphi(Z^\varepsilon_t)] - \mathbb{E}[\varphi(\bar{Z}_t)] \right| \leqslant C_T\, \varepsilon^{(\vartheta\wedge 1)/2},$$

where

$$\mathrm{d}\bar{Z}_t = \nabla_y \bar{F}(\bar{Y}_t)\bar{Z}_t \mathrm{d}t + \nabla_y G(\bar{Y}^1_t)\bar{Z}_t \mathrm{d}W^2_t + \sqrt{\overline{\tilde{F}\cdot\Gamma^*}(\bar{Y}_t)}\mathrm{d}\tilde{W}_t.$$

Main results - functional CLT

Now, we consider Case 1 in (2.5). According to Theorem 1 again, we have

$$\mathbb{E}|Y_t^\varepsilon - \bar{Y}_t^1| \leqslant C_1\big(\alpha_\varepsilon/\gamma_\varepsilon + \alpha_\varepsilon^2/(\beta_\varepsilon\gamma_\varepsilon)\big).$$

Define the normalized difference

$$Z_t^{1,\varepsilon} := \frac{Y_t^\varepsilon - \bar{Y}_t^1}{\eta_\varepsilon}.$$

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$$Z_t^{1,\varepsilon} := \frac{Y_t^\varepsilon - \bar{Y}_t^1}{\eta_\varepsilon}.$$

Then the natural choice of the deviation scale η_ε in order to observe non-trivial behavior for $Z_t^{1,\varepsilon}$ should be divided into the following three cases:

$$\begin{cases} \eta_\varepsilon = \dfrac{\alpha_\varepsilon^2}{\beta_\varepsilon\gamma_\varepsilon} \quad \text{and} \quad \lim\limits_{\varepsilon\to 0}\dfrac{\beta_\varepsilon}{\alpha_\varepsilon} = 0, & \text{Case 1.1}; \\ \eta_\varepsilon = \dfrac{\alpha_\varepsilon}{\gamma_\varepsilon} \quad \text{and} \quad \lim\limits_{\varepsilon\to 0}\dfrac{\alpha_\varepsilon}{\beta_\varepsilon} = 0, & \text{Case 1.2}; \\ \eta_\varepsilon = \dfrac{\alpha_\varepsilon}{\gamma_\varepsilon} \quad \text{and} \quad \alpha_\varepsilon = \beta_\varepsilon, & \text{Case 1.3}. \end{cases}$$

Recall that Φ is the unique solution to the Poisson equation

$$\mathscr{L}_0(x, y)\Phi(x, y) = -H(x, y).$$

Define

$$\overline{c \cdot \nabla_x \Phi}(y) := \int_{\mathbb{R}^{d_1}} c(x, y) \cdot \nabla_x \Phi(x, y) \mu^y(\mathrm{d}x);$$

$$\overline{H \cdot \Phi^*}(y) := \int_{\mathbb{R}^{d_1}} H(x, y) \cdot \Phi^*(x, y) \mu^y(\mathrm{d}x).$$

Main results - functional CLT

Theorem 3 (CLT: Case 1) [Röckner and X., 2021, CMP]

The limiting processes $\bar{Z}^1_{\ell,t}$ $(\ell = 1, 2, 3)$ for $Z^{1,\varepsilon}_t$ corresponding to Case 1.1-Case 1.3 satisfy

$$
\begin{aligned}
\mathrm{d}\bar{Z}^1_{1,t} &= \nabla_y \bar{F}_1(\bar{Y}^1_t)\bar{Z}^1_{1,t}\mathrm{d}t + \nabla_y G(\bar{Y}^1_t)\bar{Z}^1_{1,t}\mathrm{d}W^2_t + \overline{c\cdot\nabla_x\Phi}(\bar{Y}^1_t)\mathrm{d}t;\\
\mathrm{d}\bar{Z}^1_{2,t} &= \nabla_y \bar{F}_1(\bar{Y}^1_t)\bar{Z}^1_{2,t}\mathrm{d}t + \nabla_y G(\bar{Y}^1_t)\bar{Z}^1_{2,t}\mathrm{d}W^2_t + \sqrt{\overline{H\cdot\Phi^*}(\bar{Y}^1_t)}\mathrm{d}\tilde{W}_t;\\
\mathrm{d}\bar{Z}^1_{3,t} &= \nabla_y \bar{F}_1(\bar{Y}^1_t)\bar{Z}^1_{3,t}\mathrm{d}t + \nabla_y G(\bar{Y}^1_t)\bar{Z}^1_{3,t}\mathrm{d}W^2_t\\
&\quad + \overline{c\cdot\nabla_x\Phi}(\bar{Y}^1_t)\mathrm{d}t + \sqrt{\overline{H\cdot\Phi^*}(\bar{Y}^1_t)}\mathrm{d}\tilde{W}_t,
\end{aligned}
$$

where $\tilde{W}_t$ is another Brownian motion independent of W^2_t.

Main results - functional CLT

Finally, we consider Case 2 in (2.5), where homogenization already occurs even in the LLN. According to Theorem 1 (Case 2), we have

$$\mathbb{E}|Y_t^\varepsilon - \bar{Y}_t^2| \leqslant C_2\big(\alpha_\varepsilon/\gamma_\varepsilon + \alpha_\varepsilon^2/\beta_\varepsilon\big).$$

Define the normalized difference

$$Z_t^{2,\varepsilon} := \frac{Y_t^\varepsilon - \bar{Y}_t^2}{\eta_\varepsilon}.$$

Main results - functional CLT

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$$Z_t^{2,\varepsilon} := \frac{Y_t^\varepsilon - \bar{Y}_t^2}{\eta_\varepsilon}.$$

The natural choice of the derivation scale η_ε should be divided into the following three cases:

$$\begin{cases} \eta_\varepsilon = \dfrac{\alpha_\varepsilon^2}{\beta_\varepsilon} \quad \text{and} \quad \lim\limits_{\varepsilon\to0} \dfrac{\beta_\varepsilon}{\alpha_\varepsilon\gamma_\varepsilon} = 0, & \text{Case 2.1;} \\ \eta_\varepsilon = \dfrac{\alpha_\varepsilon}{\gamma_\varepsilon} \quad \text{and} \quad \lim\limits_{\varepsilon\to0} \dfrac{\alpha_\varepsilon\gamma_\varepsilon}{\beta_\varepsilon} = 0, & \text{Case 2.2;} \\ \eta_\varepsilon = \dfrac{\alpha_\varepsilon}{\gamma_\varepsilon} = \dfrac{\alpha_\varepsilon^2}{\beta_\varepsilon}, & \text{Case 2.3.} \end{cases}$$

Main results - functional CLT

Recall that

$$\mathscr{L}_0(x,y)\Phi(x,y) = -H(x,y), \ \mathscr{L}_0(x,y)\Gamma(x,y) = -\big[F(x,y) - \bar{F}_1(y)\big].$$

Let Ψ solves the following Poisson equation:

$$\mathscr{L}_0(x,y)\Psi(x,y) = -\big[c(x,y)\cdot\nabla_x\Phi(x,y) - \overline{c\cdot\nabla_x\Phi}(y)\big].$$

Denote by

$$\overline{c\cdot\nabla_x\Psi}(y) := \int_{\mathbb{R}^{d_1}} c(x,y)\cdot\nabla_x\Psi(x,y)\mu^y(\mathrm{d}x).$$

Main results - functional CLT

Theorem 4 (CLT: Case 2) [Röckner and X., 2021, CMP]

The limiting processes $\bar{Z}^2_{\ell,t}$ ($\ell = 1, 2, 3$) for $Z^{2,\varepsilon}_t$ corresponding to Case 2.1-Case 2.3 satisfy

$$\begin{aligned} \mathrm{d}\bar{Z}^2_{1,t} &= \nabla_y \bar{F}_2(\bar{Y}^2_t)\bar{Z}^2_{1,t}\mathrm{d}t + \nabla_y G(\bar{Y}^2_t)\bar{Z}^2_{1,t}\mathrm{d}W^2_t \\ &\quad + \left(\overline{c\cdot\nabla_x\Gamma} + \overline{c\cdot\nabla_x\Psi}\right)(\bar{Y}^2_t)\mathrm{d}t; \\ \mathrm{d}\bar{Z}^2_{2,t} &= \nabla_y \bar{F}_2(\bar{Y}^2_t)\bar{Z}^2_{2,t}\mathrm{d}t + \nabla_y G(\bar{Y}^2_t)\bar{Z}^2_{2,t}\mathrm{d}W^2_t + \sqrt{\overline{H\cdot\Phi^*}(\bar{Y}^2_t)}\mathrm{d}\tilde{W}_t; \\ \mathrm{d}\bar{Z}^2_{3,t} &= \nabla_y \bar{F}_2(\bar{Y}^2_t)\bar{Z}^2_{3,t}\mathrm{d}t + \nabla_y G(\bar{Y}^2_t)\bar{Z}^2_{3,t}\mathrm{d}W^2_t \\ &\quad + \left(\overline{c\cdot\nabla_x\Gamma} + \overline{c\cdot\nabla_x\Psi}\right)(\bar{Y}^2_t)\mathrm{d}t + \sqrt{\overline{H\cdot\Phi^*}(\bar{Y}^2_t)}\mathrm{d}\tilde{W}_t, \end{aligned}$$

where $\tilde{W}_t$ is another Brownian motion independent of W^2_t.

Thank You !